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AEC RESEARCH AND DEVELOPMENT REPORT

TID-4500
13th Ed.

NYO-8671
PHYSICS

APPROXIMATE SOLUTIONS OF STEADY-STATE NEUTRON TRANSPORT PROBLEMS FOR SLABS

by

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July 1, 1958

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Contract No. AT(30-1)-1480

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ABSTRACT

The neutron transport equation is differenced with respect to all but the spatial variables and the solutions of the resulting systems of ordinary differential equations are studied. For the case of elastic scattering the approximate solution is obtained and its convergence to the exact solution is proven. If very accurate approximations are required the solution may be evaluated by computing machines. It is seen that the present procedure has many advantages over direct numerical integration procedures.

For elastic scattering the procedure is just the well-known Wick-Chandrasekhar method. The solution for homogeneous slabs of finite thickness is reduced to the inversion of at most three matrices of small order. The required calculations are easily done on desk computers. A new relation between escape probability and capture fraction is also obtained.

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APPROXIMATE SOLUTIONS OF STEADY-STATE NEUTRON TRANSPORT PROBLEMS FOR SLABS

1. Introduction.

In this paper we study some methods for the approximate solution of various steady state neutron transport problems. The methods consist in replacing some, but not all, of the continuous independent variables by a discrete set of mesh points and then differencing[1] the transport equation with respect to these discrete variables. An approximating system of functional equations is obtained in this manner. In the problems treated in this paper the approximating equations are systems of ordinary differential equations. Such procedures have been used in [2,9] for unsteady problems in which case the approximating equations form hyperbolic systems of partial differential equations. The approximating procedures of this paper are closely related to methods which have been proposed for the direct numerical solution of transport problems [1,2,3,4,5].

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[1] The term "differencing" is used in a generalized sense which may include integrating or averaging the equations over a mesh interval as well as the use of numerical integration techniques.

In Part I (Sections 2-5) we consider the elastic scattering transport equation for an infinite plane slab of finite thickness and arbitrary composition. The convergence of the approximate solution, which is obtained explicitly, to the exact solution in the appropriate limit is demonstrated. It is also shown that the approximate solution is stable with respect to the growth of small errors introduced by approximation or inaccurate knowledge of the inhomogeneous source terms. The method and solution are easily modified to solve the spherically symmetric case.

In Part II (Sections 6-10) we consider the isotropic scattering mono-energetic transport equation for homogeneous slabs. The method for such problems is precisely the Wick-Chandrasekhar procedure [10,11]. While this method is discussed at some length in the literature it is believed that the present discussion simplifies its application to a variety of problems concerned with slabs of finite thickness. The problems discussed are the determination of criticality, capture fraction and escape probability. The convergence of the method is not proved but a new result relating escape probability and capture fraction is obtained.

PART I

Elastic Scattering

2. Formulation of Elastic-scattering Problems.

The steady state neutron transport equation in plane geometry is

$$(2.0a) \qquad T[\Phi(x,\mu,u)] \equiv \left\{\mu \frac{\partial}{\partial x} + \sigma_T(x,u)\right\} \Phi(x,\mu,u) - S[\Phi;\mu,u] = \\ = \mathcal{S}(x,\mu,u) \;.$$

Here: $\Phi(x,\mu,u)$ is the flux of neutrons at position x, with lethargy u, whose velocity vector makes an angle $\cos^{-1}\mu$ with the positive x-axis; $\sigma_T(x,u)$ is the total macroscopic cross-section at x for neutrons with lethargy u; $\mathcal{S}(x,\mu,u)$ is an inhomogeneous source of neutrons; S is the scattering term. For elastic scattering this term can be written as

$$(2.0b) \quad \begin{cases} S[\Phi;\mu,u] \equiv \int\limits_{u-q}^{u} g(x;u,u')\, H[\Phi;\mu,u,u']\, du' \;, \\ H[\Phi;\mu,u,u'] \equiv \int\limits_{0}^{\pi} \Phi(x,m(\mu,u-u',\theta),u')\, d\theta \;. \end{cases}$$

For a scattering element with atomic weight A the quantities to be used in (2.0b) are defined as:

$$(2.1) \quad \begin{cases} q \equiv \ln\left(\frac{A+1}{A-1}\right)^2 \;, \quad g(x;u,u') \equiv \sigma_s(x,u') \frac{(A+1)^2}{4\pi A}\, e^{u'-u} \;, \\ m(\mu,y,\theta) \equiv \mu\alpha(y) + [1-\mu^2]^{1/2}\,[1-\alpha^2(y)]^{1/2} \cos\theta \;, \\ \alpha(y) \equiv \frac{1}{2}\,[(A+1)e^{y/2} - (A-1)\, e^{-y/2}] \;. \end{cases}$$

Here $\sigma_s(x,u)$ is the macroscopic scattering cross-section at x for neutrons with lethargy u. The derivation and physical interpretation of the quantities in (2.1) are thoroughly discussed in the literature [5,6,7]. If more than one scattering element is present additional scattering terms of the form (2.0b) must be included in (2.0a). In spherically symmetric geometry the scattering term retains the above form while the differential operator in (2.0a) becomes

$$\left\{ \mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} + \sigma_T(r,u) \right\} .$$

Such problems may be treated by slightly modifying the present method, say in a manner clearly shown in [1] or [2].

The flux of neutrons entering the slab, $-a \leq x \leq a$, from the outside must be specified. If we assume no sources present in the vacuum external to this slab there can be no neutrons entering it and we have the boundary conditions

(2.2) $\Phi(-a,\mu,u) = 0, \quad 0 < \mu \leq 1 ; \quad \Phi(a,\mu,u) = 0, \quad -1 \leq \mu < 0.$

A specified non-zero flux incident on the slab can be included in the present formulation either by modifying (2.2) to include inhomogeneous terms or by including spatial Dirac δ-functions in the inhomogeneous source $\mathcal{S}(x,\mu,u)$ in (2.0a).

Since only elastic scattering is considered there can be no neutrons present with energy greater than that

of the maximum energy source neutrons. (If fission were allowed the maximum energy would be that of the fission spectrum.) If this energy is E_{max} and the lethargy variable corresponding to any energy E is defined by

$$u \equiv \ln \frac{E_{max}}{E} \quad ,$$

then $u \geq 0$ for all neutrons in the present problem. That is $\Phi(x,\mu,u) \equiv 0$ for $u < 0$ and by (2.0b), $S[\Phi;\mu,0] = 0$. Thus at the maximum energy (2.0a) simplifies and can be solved subject to the boundary conditions (2.2). The solution is

$$(2.3)\quad \Phi(x,\mu,0) = \begin{cases} \dfrac{1}{\mu}\displaystyle\int_{-a}^{x} \mathcal{S}(\xi,\mu,0)\ \exp[\frac{-1}{\mu}\int_{\xi}^{x} \sigma_T(\xi',0)d\xi']d\xi, & 0<\mu\leq 1; \\ \dfrac{1}{|\mu|}\displaystyle\int_{x}^{a} \mathcal{S}(\xi,\mu,0)\ \exp[\frac{-1}{|\mu|}\int_{x}^{\xi} \sigma_T(\xi',0)d\xi']d\xi, & -1\leq\mu<0. \end{cases}$$

If fission is present or the scattering is isotropic $S[\Phi;\mu,0] \neq 0$ and the flux of maximum energy neutrons cannot be simply obtained as above.

3. Reduction to Differential-Difference Equations.

The continuous variables (μ, u) are to be replaced by a mesh of discrete points (μ_j, u_n) at each of which the flux, $\Phi(x, \mu_j, u_n)$, is to be approximated by a function $\phi_j^n(x)$. These approximating functions are determined as a solution of a system of linear differential equations which in some sense approximate the transport equation (2.0). There is no unique procedure by which to derive systems of approximating equations and <u>a priori</u> it cannot be determined which of a class of "equivalent systems" will yield the best approximations to Φ. (Two systems are equivalent if they approximate (2.0) to the same order of magnitude in the mesh spacing.) Of course any approximating system should satisfy the conditions of consistency (approximating equations $\longrightarrow$ transport equation as $\Delta\mu, \Delta u \longrightarrow 0$), convergence $(\phi_j^n(x) \longrightarrow \Phi(x, \mu_j, u_n)$ as $\Delta\mu, \Delta u \longrightarrow 0)$ and stability ("small" changes in inhomogeneous terms produce "small" changes in the solution). These conditions are defined more precisely in Section 5 and are shown to hold for the approximating system derived in the present section.

Since, in actual computations, the limit as $\Delta\mu, \Delta u \longrightarrow 0$ cannot be attained additional conditions are frequently imposed on the approximating equations to insure that their solutions have some required physical properties. For instance, since Φ is a neutron flux, it is reasonable to

require that $\phi_j^n(x) \geq 0$ or, as Carlson [1] does, to require that the total number of nuetrons be conserved. In the present paper only the positivity condition will be considered.

The approximating equations will be derived in two steps, first differencing with respect to the angular variable, μ, and second differencing with respect to the lethargy variable, u. This procedure will exhibit two different (and not equivalent) approximating systems of functional equations. It is then easy to modify the procedure and a clear physical interpretation of the equations in each stage is made obvious. The mesh spacing is taken to be uniform only for simplicity of presentation.

3A. Angular Differencing.

The μ-interval, $-1 \leq \mu \leq 1$, is divided into $(2J-1)$ subintervals of length $\Delta\mu$ by the mesh points

$$(3.0) \qquad \mu_j = \frac{j}{|j|}\left(|j| - \tfrac{1}{2}\right)\Delta\mu, \quad 1 \leq |j| \leq J, \quad \Delta\mu = \frac{2}{2J-1} .$$

This mesh has been chosen to avoid $\mu = 0$ as a mesh point. Such a point could be included provided the equations corresponding to (3.2) in the new mesh are "centered" at $\mu_{j\pm 1/2}$ as is done in [1] and [2]. The flux and scattering term at each angular mesh point are to be approximated by the quantities

$$(3.1) \qquad \Phi(x,\mu_j,u) \approx \phi_j(x,u) , \qquad S[\Phi;\mu_j,u] \approx S_j[\phi_k;u] .$$

Replacing corresponding terms in (2.0a) by the above approximations and setting $\mu = \mu_j$ we obtain for each j:

$$(3.2) \quad T_{\Delta\mu}[\phi_j(x,u)] \equiv \left\{\mu_j \frac{\partial}{\partial x} + \sigma_T(x,u)\right\}\phi_j(x,u) - S_j[\phi_k;u] = \mathcal{S}_j(x,u),$$
$$1 \leq |j| \leq J .$$

Here we have used the obvious notation $\mathcal{S}_j(x,u) \equiv \mathcal{S}(x,\mu_j,u)$. By specifying the dependence of the approximate scattering term, $S_j[\phi_k,u]$, on the approximate fluxes, $\phi_k(x,u)$, (3.2) becomes a system of functional equations in the unknowns $\phi_j(x,u)$.

The form of S_j is determined by using a numerical integration technique in (2.0b). A straightforward procedure

is to approximate the integrand in H by a piecewise linear function:

$$\Phi(x,m(\mu_j,y,\theta),u) \approx \left[\frac{\mu_{k+1}-m(\mu_j,y,\theta)}{\Delta\mu}\right]\phi_k(x,u) + \tag{3.3}$$

$$+ \left[\frac{m(\mu_j,y,\theta)-\mu_k}{\Delta\mu}\right]\phi_{k+1}(x,u) ,$$

$$\text{when } \begin{cases} \mu_k \leq m(\mu_j,y,\theta) \leq \mu_{k+1} , \\ u-q \leq y \leq u . \end{cases}$$

Integrating this approximate integrand with respect to θ over the interval $0 \leq \theta \leq \pi$ we obtain, provided $0 \leq u-u' \leq q$,

$$H_j[\phi_k;u,u'] \equiv \sum_{|k|=1}^{J} \omega_{jk}(u-u')\phi_k(x,u') \approx H[\Phi;\mu_j,u,u'] . \tag{3.4}$$

The weights, ω_{jk}, may be derived by carrying out the indicated integrations. We do not include them here but rather list their most important properties:

$$\omega_{jk}(y) \geq 0, \quad \sum_{|k|=1}^{J} \omega_{jk}(y) = \pi, \quad \omega_{jk}(0) = \pi\delta_{jk}, \quad (0 \leq y \leq q). \tag{3.5}$$

(The last of these properties is crucial in obtaining the simple explicit solution of Section 4.) If a more accurate interpolation is used in place of (3.3) different weights are obtained in (3.4). However, these weights still satisfy the last two conditions of (3.5) and can be made non-negative by requiring that the interpolation formula be essentially of the closed type (see [8]). We assume then an approximation

of the form (3.4) with weights satisfying (3.5) and an accuracy not less than would be obtained by using linear interpolation as in (3.3). The approximate scattering term is now defined by using (3.4) in (2.0b) to get

$$
\begin{aligned}
S_j[\phi_k;u] &\equiv \int_{u-q}^{u} g(x;u,u')\, H_j[\phi_k;u,u']\, du' \quad , \\
(3.6) \qquad &= \sum_{|k|=1}^{J} \int_{u-q}^{u} g(x;u,u') \;\omega_{jk}(u-u')\, \phi_k(x,u')\, du' .
\end{aligned}
$$

Equations (3.2) and (3.6) form a system of $2J$ integro-differential equations, the first approximating system, for the determination of the $\phi_j(x,u)$, the first approximating fluxes. The boundary conditions to be imposed are, from (2.2),

$$(3.7) \qquad \phi_j(-a,u) = 0, \quad 1 \leq j \leq J; \qquad \phi_j(a,u) = 0, \quad -J \leq j \leq -1$$

As in Section 2 we may solve for $\phi_j(x,0)$ since $S_j[\phi_k,0] = 0$. The solution is exactly of the form (2.3) and we find that

$$(3.8) \qquad \phi_j(x,0) = \overline{\Phi}(x,\mu_j,0) .$$

3B. Lethargy Differencing.

The lethargy interval, $0 \leq u \leq U$, is divided into N intervals of length Δu by the mesh points

$$(3.9) \qquad u_n = n\Delta u, \quad 0 \leq n \leq N, \quad \Delta u = \frac{q}{M} \; (= \frac{1}{N} \text{ if } q = \infty).$$

With this choice of spacing the lethargy integrations in (3.6) are over M intervals (provided $u \geq q$). If more than one scattering element is present the integrations in the additional scattering terms may be over a non-integral number of u-intervals. However an interpolation procedure, as indicated in [5], could be used to approximate the integrals over fractional lethargy intervals. In practical computations the mesh (3.9) is likely to be chosen as non-uniform to eliminate the need for many lethargy points if $U/q \gg 1$ for some scatterer.

We now approximate the first approximating fluxes and scattering terms at each lethargy mesh point by

$$(3.10) \qquad \phi_j(x,u_n) \approx \phi_j^n(x), \qquad S_j[\phi_k;u_n] \approx S_j^n[\phi_k^m].$$

Replacing corresponding terms in (3.2) by the above approximations we obtain, with $u = u_n$,

$$(3.11) \quad T_{\Delta\mu}^{\Delta u}[\phi_j^n(x)] \equiv \left\{\mu_j \frac{d}{dx} + \sigma_T{}^n(x)\right\} \phi_j^n(x) - S_j^n[\phi_k^m] = \mathcal{S}_j^n(x),$$

$$\begin{cases} 1 \leq |j| \leq J \\ 0 \leq n \leq N. \end{cases}$$

Here we have used $\sigma_T^{\,n}(x) \equiv \sigma_T(x,u_n)$ and $\mathcal{S}_j^n(x) \equiv \mathcal{S}(x,\mu_j,u_n)$. The explicit form of S_j^n is determined by using numerical integration in (3.6). If the trapezoidal rule is used we obtain

$$S_j^n[\phi_k^m] = \sum_{m=n-M}^{n} \left\{ \sum_{|k|=1}^{J} \Omega_{jk}^{nm}(x)\, \phi_k^m(x) \right\}, \tag{3.12}$$

where:

$$\Omega_{jk}^{nm}(x) \equiv \varepsilon_m g^{nm}(x)\,\omega_{jk}^{n-m}\,; \quad \varepsilon_o = \varepsilon_n = \frac{\Delta u}{2},\ \varepsilon_m = \Delta u,\ m \neq 0,n;$$

$$\omega_{jk}^{n-m} \equiv \omega_{jk}([n-m]\,\Delta u)\,; \tag{3.13}$$

$$g^{nm}(x) = g(x,u_n,u_m) \equiv \sigma_s^{\,m}(x)\,\frac{(A+1)^2}{4\pi A}\,e^{-(n-m)\Delta u} .$$

If a more accurate integration formula is used (3.12) and (3.13) are modified only by a change in the definitions of the ε_m. However, by again requiring a formula of closed type the new weights are still positive. For lethargy points $u_n < q$, $n-M$ is negative. However, since there are no neutrons with negative lethargy, we set $\phi_k^m(x) \equiv 0$ for $m < 0$ and in these cases the sum (3.12) starts at $m = 0$. If hydrogen is the scatterer $q = \infty$ and the sum always starts from $m = 0$.

The system of differential equations (3.11-.12) is the second approximating system for the determination of the 2NJ approximating fluxes, $\phi_j^n(x)$. The boundary conditions are,

by (2.2) or (3.7),

(3.14) $\phi_j^n(-a) = 0, \quad 1 \leq j \leq J; \quad \phi_j^n(a) = 0, \quad -J \leq j \leq -1; \quad 0 \leq n \leq N.$

Again we may solve for the maximum energy flux, $\phi_j^o(x)$, since $S_j^o[\phi_k^m] = 0$. We find that

(3.15) $$\phi_j^o(x) = \phi_j(x,0) = \Phi(x,\mu_j,0) .$$

4. Solution of Differential-Difference Equations.

The system for the determination of the second approximating fluxes, $\phi_j^n(x)$, can be solved exactly in simple closed form. The key to this solution is the observation, from (3.5) and (3.13), that

$$\Omega_{jk}^{nn}(x) = \Delta u \, \frac{\pi}{2} \, g^{nn}(x) \, \delta_{jk} \; . \tag{4.0}$$

Then (3.12) can be written as

$$\begin{aligned} S_j^n[\phi_k^m] &= \Delta u \, \frac{\pi}{2} \, g^{nn}(x) \, \phi_j^n(x) + \bar{S}_j^n[\phi_k^m] \; , \\ \bar{S}_j^n[\phi_k^m] &= \sum_{m=n-M}^{n-1} \left\{ \sum_{|k|=1}^{J} \Omega_{jk}^{nm}(x) \, \phi_k^m(x) \right\} . \end{aligned} \tag{4.1}$$

Using this result in (3.11) yields

$$\left\{ \mu_j \frac{d}{dx} + a^n(x) \right\} \phi_j^n(x) = \bar{S}_j^n[\phi_k^m] + \mathcal{S}_j^n(x) \; , \; \begin{cases} 1 \le |j| \le J, \\ 1 \le n \le N \; . \end{cases} \tag{4.2}$$

Here a modified cross-section has been introduced by the definition

$$\begin{aligned} a^n(x) &\equiv \sigma_T{}^n(x) - \Delta u \, \frac{\pi}{2} \, g^{nn}(x) \; , \\ &= \sigma_T{}^n(x) \left[1 - \Delta u \, \frac{\sigma_s{}^n(x)}{\sigma_T{}^n(x)} \, \frac{(A+1)^2}{8A} \right] . \end{aligned} \tag{4.3}$$

From (4.1) it is seen that $\bar{S}_j^n$ is independent of $\phi_j^n(x)$. Hence for each (n,j) (4.2) is a linear first order inhomogeneous differential equation for $\phi_j^n(x)$. Boundary

conditions are given, for $j \geq 1$ at $x = -a$ and for $j \leq -1$ at $x = a$, in (3.14). The solutions are found to be

$$
(4.4)\quad \phi_j^n(x) = \begin{cases} \dfrac{1}{\mu_j} \displaystyle\int_{-a}^{x} [\bar{S}_j^n[\phi_k^m(\xi)] + \mathcal{S}_j^n(\xi)] \exp [\frac{-1}{\mu_j} \int_{\xi}^{x} \alpha^n(\xi')d\xi', & 1 \leq j \leq J ; \\ \dfrac{1}{|\mu_j|} \displaystyle\int_{x}^{a} [\bar{S}_j^n[\phi_k^m(\xi)] + \mathcal{S}_j^n(\xi)] \exp [\frac{-1}{|\mu_j|} \int_{x}^{\xi} \alpha^n(\xi')d\xi']d\xi, & -J \leq j \leq -1 . \end{cases}
$$

Starting from the maximum energy flux, $\phi_j^o(x)$ (which is given in (3.15)), we may obtain the solution for all n by repeated application of (4.1) and (4.4). If the source and medium are piecewise constant all integrations can be performed explicitly. The solution will then be composed of linear combinations of exponentials of the form: $\exp(^{-\alpha^n x}/|\mu_j|)$.

The solution (4.4) yields only positive fluxes (since all cross-sections and the source are non-negative). However, the flux coming directly from the source at any point may be exponentially growing if $\alpha^n(x)$ is negative in a neighborhood of that point. This will not be the case if, as seen from (4.3), Δu is sufficiently small. On a sequence of meshes in which $\Delta u \longrightarrow 0$ the $\alpha^n(x)$ eventually become and stay positive. In actual computations, however, a fixed mesh is

used and the above considerations suggest that we require[2]

$$(4.5) \qquad \Delta u < \max_{\substack{-a \le x \le a \\ 0 \le n \le N}} \left[\frac{\sigma_T^{\,n}(x)}{\sigma_s^{\,n}(x)} \; \frac{8A}{(A+1)^2} \right] .$$

With such a choice for Δu the exponential decay $e^{-\alpha x/|\mu|}$, rather than $e^{-\sigma_T x/|\mu|}$, may be considered as a correction for the fact that some neutrons are "scattered" straight ahead in elastic scattering and thus the total cross-section, σ_T, is too large. The best choice for Δu might be determined by examining some experimental data which measures this foreward scattering effect.

In the case of spherically symmetric geometry explicit solutions can also be obtained. The equations corresponding to (4.2) will be coupled pairwise in the angular subscript (that is ϕ_j^n and ϕ_{j-1}^n both appear) in all but one equation, say that for $j = J$. Starting with this equation the system may be solved recursively. The solution is of the form (4.4) with the $\bar{S}_j^n$ replaced by successively more complicated expressions as j decreases.

If many angular or lethargy mesh points are used the solutions (4.4) may become too complicated for direct analysis. In such cases a numerical evaluation and tabulation of the

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[2] This condition was found to be sufficient in [5] for the solvability of the difference equations considered there. With an additional restriction on the size of the spatial mesh it also guaranteed positive numerical fluxes and convergence of the difference solution.

approximate solution is easily accomplished with the aid of a computing machine. These calculations are in some sense equivalent to direct numerical integrations of the transport equation. However, the problem of evaluating expressions of the form occurring in (4.4) is straightforward and thus should have many advantages over previously proposed direct numerical procedures [1,5].

5. Consistency, Convergence and Stability.

The approximating system (3.11) is consistent with the transport equation (2.0a) if

$$(5.0)\quad \lim_{\substack{\Delta\mu\to 0\\ \Delta u\to 0}} \left| T[\psi(x,\mu,u)] - T^{\Delta u}_{\Delta\mu}[\psi(x,\mu_j,u_n)]\right|_{\substack{\mu=\mu_j\\ u=u_n}} = 0, \begin{cases} 1\le |j| \le J \\ 0\le n < N, \end{cases}$$

for all sufficiently smooth functions $\psi(x,\mu,u)$. To demonstrate this property we note that, by (2.0), (3.2), (3.6), (3.11) and (3.12),

$$T[\psi]-T^{\Delta u}_{\Delta\mu}[\psi] = (T[\psi]-T_{\Delta\mu}[\psi]) + (T_{\Delta\mu}[\psi] - T^{\Delta u}_{\Delta\mu}[\psi])$$

$$= (S[\psi] - S_j[\psi]) + (S_j[\psi] - S^n_j[\psi])$$

$$= \int_{u_n-q}^{u_n} du' g(x,u_n,u') \left\{ \int_0^\pi \psi(x,m(\mu_j,u_n-u',\theta),u')d\theta - \right.$$

$$\left. - \sum_{|k|=1}^{J} \omega_{jk}(u_n-u')\ \psi(x,\mu_k,u')\right\}$$

$$+ \sum_{|k|=1}^{J} \left\{ \int_{u_n-q}^{u_n} du' g(x,u_n,u')\ \omega_{jk}(u_n-u')\psi(x,\mu_k,u') - \right.$$

$$\left. - \sum_{m=n-M}^{n} e_m\ \Omega^{nm}_{jk}(x)\ \psi(x,\mu_k,u_n)\right\}.$$

Here we have abbreviated the previous notation in an obvious manner. Each of the curly brackets above contains the difference between some integral and an approximation to that

integral. These differences can be estimated by standard formulae. If the simplest procedures described in Section 3 are used in these approximate integrations it is easily shown that

$$(5.1) \quad |T[\psi]-T_{\Delta\mu}^{\Delta u}[\psi]| \leq (\Delta\mu)^2 \frac{1}{4} \left|\sigma_s \frac{\partial^2\psi}{\partial\mu^2}\right| + (\Delta u)^2 \frac{Jq}{6}\left|\frac{\partial^2}{\partial u^2}(q\,\omega\,\psi)\right| .$$

Here the maxima of the absolute values are to be used. If more accurate integration formulae had been used the right hand side of (5.1) would contain higher powers of $\Delta\mu$ and Δu. It is clear that (5.0) is satisfied provided (since $J = \frac{1}{\Delta\mu} + \frac{1}{2}$) the passage to the limit is such that $\Delta u^2/\Delta\mu \longrightarrow 0$. Thus the equations are consistent if the mesh spacings are related by

$$(5.2) \qquad \Delta u \leq \text{const } (\Delta\mu)^{(1/2+\varepsilon)} , \qquad \text{for any} \quad \varepsilon > 0 .$$

This is not a severe restriction and could be relaxed if a more accurate lethargy integration is used[3]. It will be shown below that (5.2) is also a sufficient condition for convergence and stability.

The approximate solution will converge to the exact solution provided

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[3] If the error in the lethargy integration is of order $(\Delta u)^n$ condition (5.2) is replaced by: $\Delta u \leq \text{const}(\Delta\mu)^{(1/n+\varepsilon)}$ for any $\varepsilon > 0$.

$$\lim_{\substack{\Delta\mu \to 0 \\ \Delta u \to 0}} |\Phi(x,\mu_j,u_n) - \phi_j^n(x)| = 0 \;, \quad \begin{cases} 1 \le |j| \le J \\ 0 \le \; n \; \le N \end{cases} .$$

To prove this we obtain and solve a system of equations for the error,

$$(5.3) \qquad e_j^n(x) \equiv \Phi(x,\mu_j,u_n) - \phi_j^n(x) \;,$$

and show that it can be made arbitrarily small as the mesh is refined. Setting $\mu = \mu_j$ and $u = u_n$ in (2.0a) and subtracting (3.11) from it yields, using (5.3),

$$(5.4) \qquad T_{\Delta\mu}^{\Delta u}[e_j^n(x)] = S[\Phi;\mu_j,u_n] - S_j^n[\Phi] \equiv \tau_j^n(x), \begin{cases} 1 \le |j| \le J \\ 0 \le \; n \; \le N \end{cases} .$$

Exactly as in (5.1) we see that

$$(5.5) \qquad |\tau_j^n(x)| \le \Delta\mu^2 \frac{1}{4} \left| \sigma_s \frac{\partial^2 \Phi}{\partial \mu^2} \right| + \Delta u^2 \frac{Jq}{6} \left| \frac{\partial^2}{\partial u^2}(g\omega\Phi) \right| \equiv \tau .$$

At the boundaries the error satisfies, from (2.2) and (3.7),

$$(5.6) \qquad e_j^n(-a) = 0, \quad 1 \le j \le J \;; \quad e_j^n(a) = 0 \;, \quad -J \le j \le -1 \;;$$
$$0 \le n \le N .$$

The error is thus determined as the solution of the system (5.4) subject to the boundary conditions (5.6). This system is formally identical with that, (3.11) and (3.14), satisfied by the fluxes $\phi_j^n(x)$ if in the latter we replace $\mathcal{S}_j^n(x)$ by $\tau_j^n(x)$. Thus by the procedure of Section 4 we obtain for the error

$$(5.7)\quad e_j^n(x) = \begin{cases} \dfrac{1}{\mu_j} \displaystyle\int_{-a}^{x} [\bar{S}_j^n[e_k^m(\xi)] + \tau_j^n(\xi)] \exp[\frac{-1}{\mu_j} \int_{\xi}^{x} a^n(\xi')d\xi']d\xi, \\ \qquad 1 \le j \le J ; \\ \dfrac{1}{|\mu_j|} \displaystyle\int_{x}^{a} [\bar{S}_j^n[e_k^m(\xi)] + \tau_j^n(\xi)]\exp[\frac{-1}{|\mu_j|} \int_{x}^{\xi} a^n(\xi')d\xi']d\xi, \\ \qquad -J \le j \le -1 . \end{cases}$$

To estimate the magnitude of this solution we introduce the quantities

$$(5.8)\quad E_m \equiv \max_{\substack{-a\le x\le a \\ -J\le k\le J}} |e_k^m(x)| , \quad \sigma_s \equiv \max_{\substack{-a\le x\le a \\ 0\le n\le N}} \sigma_s{}^n(x) , \quad a \equiv \min_{\substack{-a\le x\le a \\ 0\le n\le N}} a^n(x).$$

From (4.1) we have, since $\Omega_{jk}^{nm}(\xi) \ge 0$,

$$(5.9)\quad \begin{aligned} |\bar{S}_j^n[e_k^m(\xi)]| &\le \sum_{m=n-M}^{n-1} \sum_{|k|=1}^{J} \Omega_{jk}^{nm}(\xi) \left|e_k^m(\xi)\right| , \\ &\le \sum_{m=n-M}^{n-1} e_m g^{nm}(\xi) \left\{ \sum_{|k|=1}^{J} \omega_{jk}^{n-m} \left|e_k^m(\xi)\right| \right\}, \\ &\le \pi \sum_{m=n-M}^{n-1} e_m g^{nm}(\xi)\, E_m , \\ &\le \Delta u\, \sigma_s \frac{(A+1)^2}{4A} e^{-n\Delta u} \sum_{m=0}^{n-1} e^{m\Delta u} E_m \equiv \bar{S}_n \end{aligned}$$

Here we have used the definitions (5.8), (3.13) and the properties (3.5). Taking the absolute value in (5.7) and using (5.9), (5.8) and (5.5) yields

$$(5.10)\quad |e_j^n(x)| \leq [\bar{S}_n+\tau]\begin{cases} \dfrac{1}{\mu_j}\displaystyle\int_a^x \exp[\frac{-\alpha}{\mu_j}(x-\xi)]d\xi, & 1 \leq j \leq J\ ; \\ \dfrac{1}{|\mu_j|}\displaystyle\int_x^a \exp[\frac{-\alpha}{|\mu_j|}(\xi-x)]d\xi, & -J \leq j \leq -1\ . \end{cases}$$

The integrals may be evaluated and give, respectively

$$\frac{1}{\alpha}\,(1-e^{-\alpha(a+x)/\mu_j}) \qquad \text{and} \qquad \frac{1}{\alpha}\,(1-e^{-\alpha(a-x)/|\mu_j|})\ .$$

We note that these quantities are positive in $-a \leq x \leq a$ regardless of the sign of α. Then taking the maximum in (5.10) with respect to x and j yields

$$(5.11)\quad E_n \leq [\bar{S}_n + \tau]\ B; \qquad B \equiv \begin{cases} \dfrac{1}{\alpha} & \text{if} \quad \alpha > 0\ , \\ \dfrac{1}{\alpha}\,(1-e^{-4a\alpha/\mu}), & \text{if} \quad \alpha < 0\ . \end{cases}$$

The above inequality can be written as, using (5.9),

$$(5.12)\quad e^{n\Delta u}\, E_n \leq \Delta u\, C \sum_{m=0}^{n-1} e^{m\Delta u}\, E_m + e^{n\Delta u}\, B\,\tau\ ,$$

where

$$(5.13)\qquad C \equiv \sigma_s \frac{(A+1)^2}{4A}\, B\ .$$

Starting with $n = 1$ and applying (5.12) recursively yields the bound

$$(5.14)\quad E_n \leq \Delta u[C\, e^{-u_n}\,(1+\Delta uC)^{n-1}]\, E_o + $$

$$+ [1 + \Delta u\, C\, e^{-u_{n-1}} \cdot \frac{(1+\Delta uC)^{n-1} - e^{u_{n-1}}}{1+\ uC-e^{\Delta u}}]\, B\tau.$$

Here E_o is the maximum absolute "initial" error which by (5.3) and (3.15) vanishes indentically. However, we retain this term to aide in the stability proof.

We now take the limit in (5.14) as $\Delta\mu \longrightarrow 0$ while Δu satisfies (5.2). Then the lethargy mesh becomes and remains so fine that condition (4.5) is satisfied and $a > 0$ (as discussed in Section 4). Thus by the definitions in (5.11), for sufficiently small Δu, B is independent of the mesh spacing and the limit yields, since $\Delta u = u_n/n$,

$$(5.15) \qquad \lim_{\substack{\Delta\mu \to 0 \\ \Delta u \to 0}} E_n \leq [Ce^{(C-1)u_n} \lim_{\substack{\Delta\mu \to 0 \\ \Delta u \to 0}} (\Delta u\, E_o) + $$

$$+ [1 + \frac{C}{C-1} (e^{(C-1)u_{n-1}} - 1] B \lim_{\substack{\Delta\mu \to 0 \\ \Delta u \to 0}} \tau .$$

By (5.2), (5.3) and (5.8) the above implies convergence provided $\lim \tau = 0$. However, this is just the condition required for consistency, as may be seen from (5.0), (5.1) and (5.5), and it is satisfied since the mesh was chosen to satisfy (5.2). The convergence proof is thus complete.

The stability proof is almost identical with the convergence proof and so will only be summarized here. Let $\psi_j^n(x)$ be the solution of the system (3.11) and (3.14) when $\mathcal{S}_j^n(x)$ is replaced by $\bar{\mathcal{S}}_j^n(x)$. Then the approximating system is said to be stable if

$$(5.16) \qquad \psi_j^n(x) \longrightarrow \phi_j^n(x) \quad \text{as} \quad \bar{\mathcal{S}}_j^n(x) \longrightarrow \mathcal{S}_j^n(x) .$$

Introducing

$$(5.17) \qquad F_n \equiv \max_{\substack{-a \le x \le a \\ -J \le k \le J}} |\psi_k^n(x) - \phi_k^n(x)| ,$$

we find as above that F_n is bounded by the expression on the right hand side of (5.14) where now

$$(5.18) \quad \tau \equiv \max_{x,n,k} \left|\bar{\mathcal{S}}_k^n(x) - \mathcal{S}_k^n(x)\right| , \quad E_o \equiv B \max_{x,k} \left|\bar{\mathcal{S}}_k^o(x) - \mathcal{S}_k^o(x)\right| .$$

On any fixed mesh, the coefficients of E_o and τ are bounded and hence (5.16) follows immediately. As the mesh is refined these coefficients remain bounded, as shown in deriving (5.15), and hence the stability is demonstrated on all sequences of meshes for which the condition (4.5) becomes and remains valid. This includes all meshes satisfying (5.2) for which the approximating equations have been shown to be consistent and convergent. It can be shown[4], however, that there are sequences of meshes on which stability holds but not consistency (e.g. a sequence on which $\Delta\mu = [\Delta u]^3$).

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[4] By more careful arguments an equality can be obtained in place of (5.1). Then a condition similar to (5.2) is obtained as a necessary condition for consistency.

PART II

Monoenergetic Isotropic Scattering

6. Formulation of Isotropic Scattering Problems.

For monoenergetic (one velocity) neutrons which scatter isotropically in a homogeneous, plane, medium the steady state transport equation (2.0) becomes

$$\left\{ \mu \frac{\partial}{\partial x} + \sigma_T \right\} \Phi(x,\mu) - S[\Phi] = \mathcal{S}(x,\mu) \,, \tag{6.0a}$$

where the scattering term is now

$$S[\Phi] = \sigma_T \frac{c}{2} \int_{-1}^{1} \Phi(x,\mu) \, d\mu \quad . \tag{6.0b}$$

Here σ_T and c are constants and c is the average number of neutrons produced for each neutron which suffers a collision. Thus if no fissionable material is present

$$0 \leq c \equiv \frac{\sigma_s}{\sigma_T} \leq 1 \quad , \tag{6.1a}$$

where σ_s is the macroscopic scattering cross-section. If fission may occur then

$$0 < c \equiv \frac{\sigma_s + \nu\sigma_f}{\sigma_T} \quad , \tag{6.1b}$$

where σ_f is the macroscopic fission cross-section and ν is the average number of neutrons emitted per fission. In this case the medium may be chosen such that $c > 1$.

As in Section 2 boundary conditions are obtained by

specifying the flux of neutrons incident on the slab, $-a \leq x \leq a$. If there are known sources present in the vacuum external to the slab the boundary conditions are of the form

$$(6.2) \qquad \Phi(-a,\mu) = \ell(\mu), \quad 0 < \mu \leq 1; \quad \Phi(a,\mu) = r(\mu), \; -1 \leq \mu < 0 \; .$$

Here $\ell(\mu)$ and $r(\mu)$ are given functions which determine the incident angular flux distribution from the left and right respectively.

The above formulation includes a variety of special cases which are of interest in reactor theory. We shall formulate below three of the more important problems. The approximate solution of these problems is considered in Sections 8-10.

6A. Reactor Criticality.

Let the slab be composed of fissionable material for which $c > 1$.[5] Let there be no inhomogeneous sources of neutrons present; that is

$$\mathcal{S}(x,\mu) \equiv 0 \,, \qquad \ell(\mu) \equiv r(\mu) \equiv 0 \,. \tag{6.3}$$

The criticality problem is to find the relationship between c, σ_T and a for which (6.0,.2,.3) has a finite non-zero solution which is of one sign, say positive. This is an eigenvalue problem in which any two of the quantities (c, σ_T, a) may be given and the third is then the eigenvalue to be determined such that the corresponding eigenfunction $\Phi(x,\mu) \geq 0$.

Physically the problem is to adjust the material constants c and σ_T with the thickness of the slab, $2a$, so that the number of neutrons escaping from the slab is just equal to the excess of those produced over those absorbed in the slab.

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[5] Physically it is obvious that more than one neutron must be produced, on the average, per collision to sustain a chain reaction.

6B. Capture Fraction.

Let $c < 1$, σ_T and a be given and assume no distributed sources present in the slab; that is

$$\mathcal{S}(x,\mu) \equiv 0 \quad (6.4)$$

Let the incident flux be specified, as in (6.2). Then the number of neutrons entering the slab per unit area is

$$\begin{aligned} N_e &\equiv -\int_{-1}^{0} \mu\, r(\mu)\, d\mu + \int_{0}^{1} \mu\, \ell(\mu)\, d\mu\,, \\ &= \int_{0}^{1} \mu\, [\ell(\mu) + r(-\mu)]\, d\mu\,. \end{aligned} \quad (6.5a)$$

The number of neutrons captured in a cylinder of unit cross-sectional area through the slab and normal to its sides is

$$N_c \equiv \sigma_T(1-c) \int_{-a}^{a} dx \int_{-1}^{1} d\mu\, \Phi(x,\mu)\,, \quad (6.5b)$$

where $\Phi(x,\mu)$ is the solution of (6.0,.2,.4). Then the fraction of neutrons captured in the slab is

$$F = \frac{N_c}{N_e}\,. \quad (6.6)$$

The variation of F with c, σ_T, a and the incident flux distribution is desired.

6C. Escape Probability.

Let no neutrons enter the slab from external sources; then

$$\ell(\mu) \equiv r(\mu) \equiv 0 \quad . \tag{6.7}$$

The number of neutrons injected by the prescribed distributed source, $\mathcal{S}(x,\mu)$, into a cylinder of unit cross-sectional area through the slab and normal to its sides is

$$N_o \equiv \int_{-a}^{a} dx \int_{-1}^{1} d\mu\, \mathcal{S}(x,\mu) \quad . \tag{6.8a}$$

The number of neutrons escaping from a unit area on the sides of the slab, $x = \pm a$, is

$$\begin{aligned} N_x &\equiv -\int_{-1}^{0} \mu\, \Phi(-a,\mu)\, d\mu + \int_{0}^{1} \mu\, \Phi(a,\mu)\, d\mu \;, \\ &= \int_{0}^{1} \mu\, [\Phi(a,\mu) + \Phi(-a,-\mu)]\, d\mu \quad . \end{aligned} \tag{6.8b}$$

Here $\Phi(x,\mu)$ is the solution of (6.0,.2,.7). The fraction of source neutrons which escape from the slab, or the escape probability, is then

$$P = \frac{N_x}{N_o} \quad . \tag{6.9}$$

The variation of P with the given parameters $c < 1$, σ_T and a is desired. In escape probability problems the source is usually taken to be independent of position and frequently isotropic.

7. Approximating Systems.

As in Section 3 of Part I we shall replace the continuous variable μ by a set of $2J$ mesh points, μ_j. This procedure, when applied to the present transport problems, is just the Wick-Chandrasekhar method [10,11]. Most applications of this method which appear in the literature are to infinite or semi-infinite regions. However, we consider here only slabs of finite thickness. In addition the present treatment of the method is somewhat more general than that usually given and the resulting algebraic problems are reduced to systems or matrices of order J rather than of order $2J$. In fact the solution of a large class of problems is reduced to the inversion of at most two or three matrices of order J.

The $2J$ mesh points are required to satisfy

$$(7.0) \qquad 0 < \mu_1 < \mu_2 < \dots < \mu_J \leq 1 \; ; \quad \mu_{-j} = -\mu_j \, , \quad 1 \leq j \leq J.$$

At each point of this mesh the flux and scattering term are approximated by

$$(7.1) \qquad \Phi(x,\mu_j) \approx \phi_j(x) \, , \qquad S[\Phi(x,\mu_j)] \approx S[\phi_k(x)] \, .$$

Setting $\mu = \mu_j$ in (6.0a) and using these approximations gives

$$(7.2a) \quad \left\{ \mu_j \frac{d}{dx} + \sigma_T \right\} \phi_j(x) - S[\phi_k(x)] = \mathcal{S}_j(x), \quad 1 \leq |j| \leq J.$$

Here $\mathcal{S}_j(x) \equiv \mathcal{S}(x,\mu_j)$ and the form of the approximate

scattering term is obtained by using a numerical quadrature formula in (6.0b). By the symmetry of the mesh such an approximation can be written as

$$(7.2b) \qquad S[\phi_k(x)] \equiv \sigma_T \frac{c}{2} \sum_{k=1}^{J} \omega_k \, [\phi_k(x) + \phi_{-k}(x)] \; .$$

The ω_k are the weights for a numerical integration over the interval $0 \leq \mu \leq 1$ using the points $\mu_j > 0$ of (7.0). If we require that the formula be exact for polynomials in μ of degree $\leq N$ these weights must satisfy

$$(7.3) \qquad \omega_k > 0 \; ; \; \sum_{k=1}^{J} \omega_k \, \mu_k^n = \frac{1}{n+1} \, , \qquad n = 0,1,2,\ldots,N.$$

By a proper choice of the μ_j and ω_j we may have $N = 2J-1$, but no greater [8,10]. For the present analysis we require only $N = 1$.

The boundary conditions (6.2) are replaced by

$$(7.4) \qquad \phi_j(-a) = \ell_j \, , \;\; 1 \leq j \leq J; \;\; \phi_j(a) = r_j \, , \; -J \leq j \leq -1;$$

where $\ell_j \equiv \ell(\mu_j)$ and $r_j \equiv r(\mu_j)$. Equations (7.2) and (7.4) are the approximating systems whose solutions are sought.

For the determination and analysis of these solutions it is convenient to formulate the approximating system in matrix notation. For this purpose we introduce: the J-dimensional column vectors

$$(7.5a)\quad \phi_{\underline{+}}(x) \equiv \begin{pmatrix} \phi_{\underline{+}1}(x) \\ \vdots \\ \phi_{\underline{+}J}(x) \end{pmatrix}, \quad \mathcal{S}_{\underline{+}}(x) \equiv \begin{pmatrix} \mathcal{S}_{\underline{+}1}(x) \\ \vdots \\ \mathcal{S}_{\underline{+}J}(x) \end{pmatrix}, \quad L \equiv \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_J \end{pmatrix}, \quad R \equiv \begin{pmatrix} r_{-1} \\ \vdots \\ r_{-J} \end{pmatrix};$$

the J-order square matrices

$$(7.5b)\quad m \equiv \begin{pmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_J \end{pmatrix}, \quad \Omega \equiv \begin{pmatrix} \omega_1 & \omega_2 & \cdots & \omega_J \\ \omega_1 & \omega_2 & \cdots & \omega_J \\ \vdots & \vdots & & \vdots \\ \omega_1 & \omega_2 & & \omega_J \end{pmatrix}, \quad \begin{cases} p \equiv m^{-1} - q, \\ q \equiv \frac{c}{2}\, m^{-1}\, \Omega; \end{cases}$$

the 2J-dimensional column vectors

$$(7.5c)\qquad \phi(x) \equiv \begin{pmatrix} \phi_+(x) \\ \\ \phi_-(x) \end{pmatrix}, \qquad \mathcal{S}(x) \equiv \begin{pmatrix} \mathcal{S}_+(x) \\ \\ \mathcal{S}_-(x) \end{pmatrix};$$

and the 2J-order square matrices:

$$(7.5d)\qquad M \equiv \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \qquad A \equiv \begin{pmatrix} p & -q \\ q & -p \end{pmatrix}.$$

Using these definitions in (7.2) we obtain the matrix equation

$$(7.6)\qquad \frac{d}{dx}\,\phi(x) + \sigma_T\, A\, \phi(x) = M^{-1}\mathcal{S}(x)\ .$$

The boundary conditions (7.4) become

$$(7.7)\qquad \phi_+(-a) = L\ , \qquad \phi_-(a) = R\ .$$

By using the same numerical integration formula in

(6.5) and (6.8) as is used to obtain (7.2b) we may define approximations to the capture fraction, $f_J \approx F$, and escape probability $p_J \approx P$, which are "consistent" with the approximation of the flux. To employ the notation of (7.5) in these quantities we introduce the J-order row vector, which is a row of Ω,

$$\underrightarrow{\omega} \equiv (\omega_1, \omega_2, \ldots, \omega_J) \; . \tag{7.8}$$

The quantities (6.5) and (6.8) are then approximated by

(7.9)

$$\text{a)}\quad N_e \approx n_e \equiv \underrightarrow{\omega}\, m[L+R] \;,$$

$$\text{b)}\quad N_c \approx n_c \equiv \sigma_T(1-c)\, \underrightarrow{\omega} \int_{-a}^{a} [\phi_+(x) + \phi_-(x)]\, dx \;,$$

$$\text{c)}\quad N_o \approx n_o \equiv \underrightarrow{\omega} \int_{-a}^{a} [\mathscr{J}_+(x) + \mathscr{J}_-(x)]\, dx \;,$$

$$\text{d)}\quad N_x \approx n_x \equiv \underrightarrow{\omega}\; m[\phi_+(a) + \phi_-(-a)] \; .$$

The approximate capture fraction and escape probability are given by

$$f_J \equiv \frac{n_c}{n_e} \;, \qquad p_J \equiv \frac{n_x}{n_o} \; . \tag{7.10}$$

8. Analysis of Formal Solutions.

The system (7.6) is a system of first order ordinary differential equations with constant coefficients. Solutions of such a system can be obtained, in principle, by the application of well known formal procedures [12]. We shall employ the methods of matrix calculus [13]. Thus we introduce

$$e^{-Ay} \equiv B(y) \equiv \begin{pmatrix} B_{11}(y) & B_{12}(y) \\ B_{21}(y) & B_{22}(y) \end{pmatrix}, \tag{8.0}$$

where $B(y)$ is a 2J-order matrix and the $B_{ij}(y)$ are J-order matrices. The general solution of (7.6) can then be written as

$$\phi(x) = B(y)\xi + \frac{1}{\sigma_T} \int_0^y B(y-y')\, M^{-1} \mathscr{A}(y')\, dy' \, . \tag{8.1}$$

Here we have introduced the dimensionless variable

$$y \equiv \sigma_T\, x \, , \quad -b \leq y \leq b \, , \quad b \equiv \sigma_T\, a \, , \tag{8.2}$$

which measures distance in units of the total mean-free-path, $1/\sigma_T$, of the slab material. The 2J-dimensional column vector

$$\xi \equiv \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}, \qquad \xi_{\pm} \equiv \begin{pmatrix} \xi_{\pm 1} \\ \vdots \\ \xi_{\pm J} \end{pmatrix}, \tag{8.3}$$

is the flux at the center of the slab and can be determined by using (8.1) in the boundary conditions (7.7). However,

before treating particular problems we shall derive some properties of the matrices A and $B(y)$ which are required for the applications.

Let the powers of A be denoted by

$$(8.4) \qquad A^n \equiv \begin{pmatrix} A_{11}^{(n)} & A_{12}^{(n)} \\ A_{21}^{(n)} & A_{22}^{(n)} \end{pmatrix},$$

where the $A_{ij}^{(n)}$ are J-order matrices. Then we have

<u>Lemma I</u>: $\quad A_{11}^{(n)} = (-1)^n A_{22}^{(n)}, \quad A_{12}^{(n)} = (-1)^n A_{21}^{(n)}$.

<u>Proof</u>. Since $A^o = I$, the 2J-order identity, the result is true for $n = 0$. (Also by the definition (7.5d) of A it is true for $n = 1$.) For an induction we assume the lemma true up to some arbitrary n. Then

$$(8.5) \qquad A^{n+1} = A\,A^n = \begin{pmatrix} [pA_{11}^{(n)} - qA_{21}^{(n)}] & [pA_{12}^{(n)} - qA_{22}^{(n)}] \\ [qA_{11}^{(n)} - pA_{21}^{(n)}] & [qA_{12}^{(n)} - pA_{22}^{(n)}] \end{pmatrix},$$

and by means of the inductive hypothesis

$$A_{12}^{(n+1)} \equiv pA_{12}^{(n)} - qA_{22}^{(n)} = (-1)^n [pA_{21}^{(n)} - qA_{11}^{(n)}] = (-1)^{n+1} A_{21}^{(n+1)}.$$

Similarly it is seen that $A_{11}^{(n+1)} = (-1)^{n+1} A_{22}^{(n+1)}$ and the proof is completed.

From the power series which defines the exponential of a matrix and definitions (8.0) and (8.4) we have

$$B_{ij}(y) = \sum_{n=0}^{\infty} \frac{(-y)^n}{n!} A_{ij}^{(n)} . \tag{8.6}$$

Using Lemma I and these expressions we obtain

<u>Lemma II</u>: $B_{11}(y) = B_{22}(-y)$, $B_{12}(y) = B_{21}(-y)$.

By means of this lemma the evaluation of $B(y)$ is reduced to the evaluation of only two J-order matrices.

Another useful result is contained in

<u>Lemma III</u>: <u>For all</u> $n \geq 1$,

$$A_{11}^{(n)} \pm A_{12}^{(n)} = [p \mp (-1)^{n-1}q]\,[p \mp (-1)^{n-2}q] \ldots [p \mp q].$$

<u>Proof</u>. From the definitions (7.5d) and (8.4) the lemma is true for $n = 1$. For an induction we assume it to be true up to n. Then from (8.5)

$$A_{11}^{(n+1)} \pm A_{12}^{(n+1)} = p\,[A_{11}^{(n)} \pm A_{12}^{(n)}] \mp q\,[A_{22}^{(n)} \pm A_{21}^{(n)}] ,$$

$$= [p \mp (-1)^n\, q]\,[A_{11}^{(n)} \pm A_{12}^{(n)}] ,$$

where we have used Lemma I. The lemma now follows by an application of the inductive hypothesis.

Using Lemma III and the definitions (8.6) we may write

$$B_{11}(y) \pm B_{12}(y) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} [(p \pm q)(p \mp q)]^n - \tag{8.7}$$

$$- (p \mp q) \sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!} [(p \pm q)(p \mp q)]^n .$$

As will be shown in the next section it is the above two

J-order matrices which determine the solution of all problems with homogeneous distributed sources. To obtain a simpler representation of these matrices we must examine the matrix

$$(8.8) \qquad K^2 \equiv (p+q)(p-q) = m^{-2}\,(I - c\,\Omega)\,.$$

Here we have used the definition (7.5b) and the justification for introducing the square of a matrix, K, is as follows: Assume K^2 to have simple elementary divisors (or what is sufficient, that its eigenvalues are distinct; see Theorem I). Then there exists a diagonalizing similarity transformation [13]

$$(8.9) \qquad P^{-1} K^2 P = \Lambda \equiv (\lambda_i\,\delta_{ij})\,.$$

If we define

$$(8.10) \qquad k_i \equiv \sqrt{\lambda_i}\,, \qquad \Lambda^{1/2} \equiv (k_i\,\delta_{ij})\,,$$

then the matrix K may be chosen as

$$(8.11) \qquad K = P\,\Lambda^{1/2}\,P^{-1}\,.$$

This is a standard procedure for defining the "square-root" of a matrix. The columns of the diagonalizing matrix P are proportional to the eigenvectors of K^2.

An analysis of the eigenvectors and eigenvalues of K^2 is contained in

<u>Theorem I:</u> (a) <u>The eigenvector</u> $u \equiv \begin{pmatrix} u_1 \\ \vdots \\ u_J \end{pmatrix}$ <u>belonging to an eigenvalue</u> λ <u>of</u> K^2 <u>has components</u>

(8.12) $$u_j = \frac{\text{const.}}{1-\lambda\,\mu_j^2} \;.$$

(b) <u>The eigenvalues</u> λ_j <u>of</u> K^2 <u>are the</u> J <u>roots of</u>

(8.13) $$F_J(\lambda) \equiv \sum_{k=1}^{J} \frac{\omega_k}{1-\lambda\mu_k^2} = \frac{1}{c} \;.$$

(c) <u>The eigenvalues</u> λ_j <u>are real and distinct and lie in the intervals:</u>

(8.14) $$-\infty < \lambda_J < \frac{1}{\mu_J^2} < \lambda_{J-1} < \frac{1}{\mu_{J-1}^2} < \dots < \frac{1}{\mu_{j+1}^2} < \lambda_j < \frac{1}{\mu_j^2} < \dots < \lambda_1 < \frac{1}{\mu_1^2}$$

(d) <u>All eigenvalues are monotone decreasing functions of</u> c <u>and the smallest one has the values:</u>

(8.15) $$\lambda_J = \begin{cases} \frac{1}{\mu_J^2}, & c = 0 \\ 0, & c = 1 \\ -\infty, & c = \infty \end{cases}$$

<u>Proof:</u> An eigenvector u and its corresponding eigenvalue λ must satisfy

(8.16) $$K^2\, u = \lambda\, u \;.$$

From the definitions (8.8) and (7.5b) we find that the j-th component of (8.16) is

$$\mu_j^{-2}\,(u_j - c\sum_{k=1}^{J} \omega_k\, u_k) = \lambda\, u_j \;,$$

and thus

$$u_j = \frac{c \sum_{k=1}^{J} \omega_k u_k}{1-\lambda \mu_j^2} \ .$$

Since the numerator is independent of j part (a) of the theorem follows. Using the components (8.12) of an eigenvector in (8.16) we find, as above, that the corresponding eigenvalue must satisfy (8.13). This concludes the proof of part (b).

To examine the roots of (8.13) we observe, by (7.3), that $\omega_k > 0$ and hence the k-th term in the sum is a piecewise-monotone increasing function of λ with an infinite discontinuity at $\lambda = \mu_k^{-2}$. The function $F_J(\lambda)$ is thus composed of $J+1$ branches as indicated in Figure 1. Since $\sum_{k=1}^{J} \omega_k = 1$, by (7.3), we have $F_J(0) = 1$. The roots

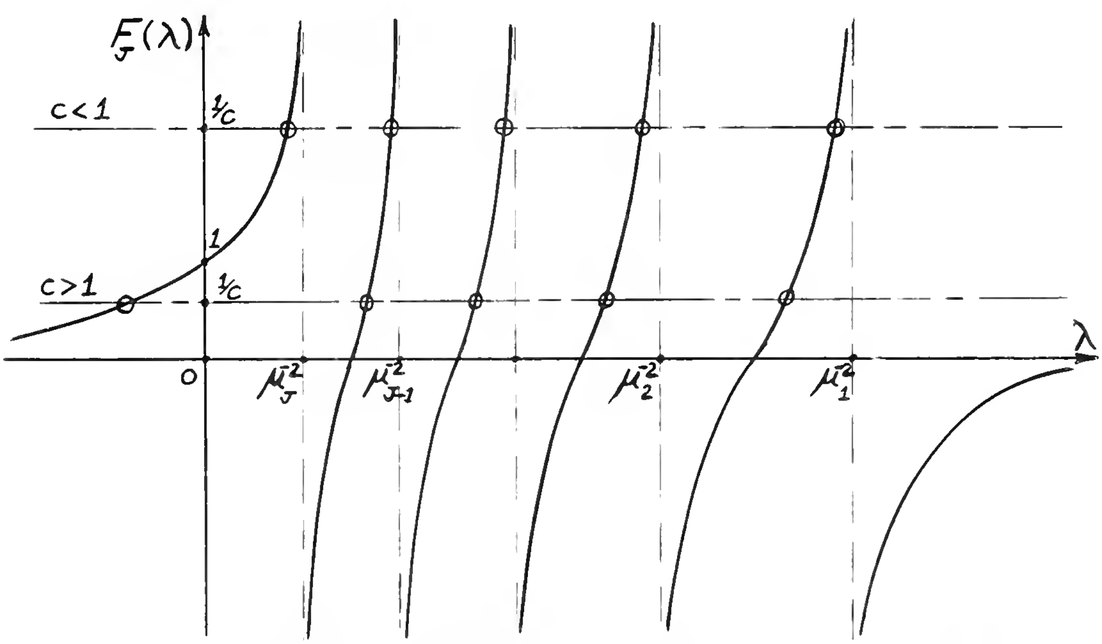

Figure 1. Schematic graph of $F_J(\lambda)$ with $J = 5$.

of (8.13) are the intersections of this curve with the horizontal line with ordinate $1/c$. Parts (c) and (d) of the theorem are immediately obvious from Figure 1 and the proof is complete.

From the definitions (8.8) and (7.5b) we have

$$(8.16) \qquad (p+q) = m^{-1}, \qquad (p-q) = m\,K^2, \qquad (p-q)(p+q) = m\,K^2\,m^{-1}.$$

Using these expressions and (8.8) in (8.7) we obtain

$$(8.17) \qquad \begin{aligned} &a) \quad B_{11}(y)+B_{12}(y) = [\cosh(Ky) - (mK)\sinh(Ky)], \\ &b) \quad B_{11}(y)-B_{12}(y) = m[\cosh(Ky) - (Km)^{-1}\sinh(Ky)]m^{-1}. \end{aligned}$$

The definitions of the hyperbolic functions of a matrix are of course the series expansions. In (8.17b) we have used K^{-1} which exists only if $c \neq 1$. The explicit dependence on y of the above matrices may be determined by using (8.10-.11) in (8.17) to obtain

$$(8.18) \qquad \begin{aligned} &a) \quad B_{11}(y)+B_{12}(y) = P[(\delta_{ij}\cosh k_j y) \\ &\qquad\qquad - (P^{-1}mP)(\delta_{ij}k_j \sinh k_j y)]\,P^{-1}, \\ &b) \quad B_{11}(y)-B_{12}(y) = (mP)\,[(\delta_{ij}\cosh k_j y) - \\ &\qquad\qquad - (P^{-1}\,m^{-1}\,P)(\delta_{ij}\,\frac{1}{k_j}\sinh k_j y)](mP)^{-1}. \end{aligned}$$

Another matrix of interest is A^{-1}. To obtain this matrix we first observe that by (7.5b) and (7.3) with $n = 0$:

$$\Omega^2 = \Omega .$$

Then from the definition (7.5d) of A we can easily verify that

$$(8.19)\qquad A^{-1} = \begin{pmatrix} r & s \\ -s & -r \end{pmatrix} ; \quad \begin{cases} r \equiv m - s , \\ s \equiv \dfrac{c}{2(c-1)} \Omega m ; \end{cases}$$

provided $c \neq 1$. The singular case, $c = 1$, will not be treated here in any detail as the present results hold for c arbitrarily close to one. However this exceptional case offers no formal difficulties and indeed is the first problem studied, for semi-infinite slabs, by Chandrasekhar [10].

9. General Solution for Homogeneous Sources.

The general problem formulated in (7.6-.7) can be solved explicitly if the source, $\mathcal{S}(x)$, is independent of position. We consider here such problems in which

$$\mathcal{S}(x) \equiv \mathcal{S} (= \text{a constant vector}) . \tag{9.0}$$

The general solution (8.1) now becomes

$$\phi(x) = B(y)\,\xi + \frac{1}{\sigma_T}\int_0^y B(y-y')\,dy'\,M^{-1}\mathcal{S} . \tag{9.1}$$

From the definition (8.0) of $B(y)$ we have

$$\int^y B(-y')\,dy' = A^{-1}B(-y) = B(-y)\,A^{-1} , \tag{9.2}$$

since $B(y)$ is a power series in A. Using (9.2) in (9.1) we may write the solution as

$$\begin{aligned} \phi(x) &= B(y)\,\xi + \frac{1}{\sigma_T}[I-B(y)]\,A^{-1}M^{-1}\mathcal{S} , \\ &= B(y)\,(\xi-\eta) + \eta , \end{aligned} \tag{9.3}$$

where we have introduced the 2J-dimensional vector

$$\eta \equiv \begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix} \equiv \frac{1}{\sigma_T}A^{-1}M^{-1}\mathcal{S} . \tag{9.4}$$

The boundary conditions (7.7), from which ξ is to be determined, become by (9.3)

$$\begin{aligned} \phi_+(-a) &= B_{11}(-b)(\xi_+-\eta_+) + B_{12}(-b)(\xi_--\eta_-) + \eta_+ = L , \\ \phi_-(a) &= B_{21}(b)(\xi_+-\eta_+) + B_{22}(b)(\xi_--\eta_-) + \eta_- = R . \end{aligned} \tag{9.5}$$

Using Lemma II the solution of this system can be written as

$$(9.6)\quad \xi_{\pm} = \eta_{\pm} + \frac{1}{2}\,[B_{22}(b)+B_{21}(b)]^{-1}[(L+R)-(\eta_{+}+\eta_{-})] \pm \pm \frac{1}{2}\,[B_{22}(b)-B_{21}(b)]^{-1}\,[(L-R)-(\eta_{+}-\eta_{-})]\ .$$

The solution (9.3) is thus determined, and by another application of Lemma II and (9.6) it may be written explicitly as

$$(9.7)\quad \begin{aligned} \phi_{\pm}(x) = \eta_{\pm} &+ \tfrac{1}{2}[B_{22}(\mp y)+B_{21}(\mp y)][B_{22}(b)+B_{21}(b)]^{-1}\times \\ &\times[(L+R)-(\eta_{+}+\eta_{-})] \pm \\ &\pm \tfrac{1}{2}[B_{22}(\mp y)-B_{21}(\mp y)][B_{22}(b)-B_{21}(b)]^{-1}\times \\ &\times[(L-R)-(\eta_{+}-\eta_{-})]\ . \end{aligned}$$

As mentioned in the previous section we see that the solution depends only on the matrices given in (8.17) or (8.18) (recalling Lemma II). The only matrices which are not given explicitly are P^{-1}, to be used in (8.18), and $[B_{22}(b)\pm B_{21}(b)]^{-1}$. These are matrices of order J and, for a particular problem, are independent of the spatial variable. For many problems of interest, including isotropic sources, $L-R = \eta_{+}-\eta_{-} = 0$ and hence $[B_{22}(b)-B_{21}(b)]^{-1}$ is not required.

The general solution (9.7) may be simplified to study a variety of special problems. In the next section we consider those problems formulated in Sections 6A, B and C. The study of the reflection and transmission matrices of slabs, which are of more interest in astrophysical problems [10], offers no difficulty but will be omitted here.

10. Special Problems.

10A. Determination of Criticality.

An approximate solution of the criticality problem formulated in Section 6A is obtained by using (6.3) in (7.5a). Then we have

$$\mathcal{S}(x) \equiv 0 , \qquad L \equiv R \equiv 0 , \tag{10.1}$$

and the formal solution (9.1) becomes

$$\phi(x) = B(y)\, \xi . \tag{10.2}$$

The boundary conditions (7.7) are, by (10.1) and Lemma II,

$$\begin{cases} B_{22}(b)\, \xi_+ + B_{21}(b)\, \xi_- = 0 , \\ B_{21}(b)\, \xi_+ + B_{22}(b)\, \xi_- = 0 . \end{cases} \tag{10.3}$$

These equations are homogeneous, in contrast to those of (9.5), and have a solution only if the coefficient matrix is singular, that is

$$\begin{vmatrix} B_{22}(b) & B_{21}(b) \\ B_{21}(b) & B_{22}(b) \end{vmatrix} = 0 . \tag{10.4}$$

The solution of Section 9 is thus not applicable and we are faced with the problem of finding values of c and $b \equiv \sigma_T a$ for which (10.4) is satisfied.

However, the above condition may be simplified by observing that the solution of interest must have the symmetry property

$$\phi_+(x) = \phi_-(-x) . \tag{10.5}$$

This is physically obvious for the solution of the exact problem from the symmetry of that problem. More precisely if $\Phi(x,\mu)$ is an eigenfunction then so is $\Phi(-x,-\mu)$, as may be verified by substitution into (6.0). The symmetry of the exact solution is then a consequence of the uniqueness of the eigenfunctions. A similar result is also true of the eigenvectors, $\phi(x)$, of the approximating system, from which (10.5) follows. From (10.5) we have in particular

$$\phi_+(0) = \phi_-(0) ,$$

and since $B(0) = I$ we deduce from (10.2) that

$$\xi_+ = \xi_- . \tag{10.6}$$

The boundary conditions (10.3) are now reduced to

$$[B_{22}(b) + B_{21}(b)]\ \xi_+ = 0 , \tag{10.7}$$

a homogeneous system of order J.

Non-trivial solutions of (10.7) exist provided that

$$|B_{22}(b) + B_{21}(b)| = 0 . \tag{10.8}$$

Equation (10.8) is called the criticality equation. It may be determined explicitly by using Lemma II and (8.18a). If $\cosh k_j b \neq 0$ for all j the criticality equation becomes

$$|I + (P^{-1}mP)(\delta_{ij}k_j \tanh k_j b)| = 0 . \tag{10.9}$$

For each root, (c,b), of (10.9) ξ_+ is determined, to within a scalar factor, as a solution of (10.7). The desired solution is that for which $\phi_+(x) \geq 0$ in $-a \leq x \leq a$. From (10.2), Lemma II, and (10.6) the solution is found to be, formally,

$$\phi_-(-x) = \phi_+(x) = [B_{22}(-y)+B_{21}(-y)]\ \xi_+ . \tag{10.10}$$

10B. Calculation of Capture Fraction.

The solution of the capture fraction problem formulated in Section 6B is approximated by using (6.4) in (7.5a). The solution of Section 9 now applies with $\delta \equiv 0$. Then by (9.4), $\eta \equiv 0$, and from (9.7) we obtain

$$\phi_+(x)+\phi_-(x) = \tfrac{1}{2}[B_{22}(-y)+B_{21}(-y)+B_{22}(y)+B_{21}(y)]\times$$
$$\times[B_{22}(b)+B_{21}(b)]^{-1}\ (L+R)\ .$$

This expression may be simplified by using Lemma II and (8.17a) to yield

(10.11) $$\phi_+(x)+\phi_-(x) = [\cosh Ky][B_{22}(b)+B_{21}(b)]^{-1}\ (L+R)\ .$$

Recalling that $y = \sigma_T\, x$ we obtain

(10.12) $$\int_{-a}^{a} [\phi_+(x)+\phi_-(x)]\, dx = \frac{2}{\sigma_T} K^{-1}[\sinh Kb][B_{22}(b)+B_{21}(b)]^{-1}(L+R).$$

For simplicity we introduce the notation

(10.13) $$Q \equiv K^{-1}\ [\sinh Kb]\ [B_{22}(b)+B_{21}(b)]^{-1}\ ,$$

and use it in (10.12) and (7.9b) to obtain

(10.14) $$n_c = 2(1-c)\ \underrightarrow{\omega}\ Q(L+R)\ .$$

Then from (7.9a) and (7.10) the capture fraction is found to be

(10.15) $$f_J = 2(1-c)\ \frac{\underrightarrow{\omega}\ Q(L+R)}{\underrightarrow{\omega}\ m(L+R)}\ .$$

10C. Calculation of Escape Probability.

The escape probability problem formulated in Section 6C is approximated by using (6.7) in (7.5a) to get $L \equiv R \equiv 0$. Taking the distributed source to be independent of position the solution of Section 9 applies and from it we obtain

$$\phi_+(a)+\phi_-(-a) = (\eta_+ + \eta_-) - [B_{22}(-b)+B_{21}(-b)][B_{22}(b)+B_{21}(b)]^{-1}(\eta_+ + \eta_-).$$

Using Lemma II and (8.17a) this expression may be written as

$$\phi_+(a)+\phi_-(-a) = 2mK[\sinh Kb][B_{22}(b)+B_{21}(b)]^{-1}\ (\eta_+ + \eta_-),$$

$$(10.16) \qquad = 2m\ K^2\ Q\ (\eta_+ + \eta_-),$$

where we have used the definition (10.13). From (9.4), (8.8) and (8.19) we obtain

$$(10.17) \qquad \begin{aligned} \eta_+ + \eta_- &= \frac{1}{\sigma_T}\left(I - \frac{c}{c-1}\,\Omega\right)(\mathcal{S}_+ + \mathcal{S}_-), \\ &= \frac{1}{\sigma_T}\,(m^2\ K^2)^{-1}\ (\mathcal{S}_+ + \mathcal{S}_-). \end{aligned}$$

Here we have used the easily verified result:
$(I - c\ \Omega)^{-1} = (I - \frac{c}{c-1}\ \Omega)$. Using (10.17) and (10.16) in (7.9d) we have

$$(10.18) \qquad n_X = \frac{2}{\sigma_T}\ \underset{\rightarrow}{\omega}\ (m^2\ K^2)\ Q(m^2\ K^2)^{-1}\ (\mathcal{S}_+ + \mathcal{S}_-).$$

Since the source was assumed independent of position (7.9c) gives

(10.19) $$n_o = 2a \underset{\rightarrow}{\omega} (\mathcal{S}_+ + \mathcal{S}_-) ,$$

and the escape probability is, from (7.10)

(10.20) $$p_J = \frac{1}{\sigma_T a} \frac{\underset{\rightarrow}{\omega} (m^2K^2)Q(m^2K^2)^{-1}(\mathcal{S}_+ + \mathcal{S}_-)}{\underset{\rightarrow}{\omega} (\mathcal{S}_+ + \mathcal{S}_-)} .$$

This form of p_J has been derived to show some formal relation with the expression (10.15) for f_J. A better representation for computations is derived in the next section; equation (10.26).

10D. A Relation between Escape Probability and Capture Fraction.

An interesting relation between the escape probability and capture fraction can be obtained when the angular dependence of the respective source terms are related by

$$(10.21) \qquad (\mathcal{S}_+ + \mathcal{S}_-) = \text{const.} \times (L+R) \equiv S \; .$$

In terms of the continuous source functions this implies that

$$(10.22) \qquad [\mathcal{S}(\mu)+\mathcal{S}(-\mu)] = \text{const.}[\ell(\mu)+r(-\mu)] \equiv s(\mu), \quad 0 \leq \mu \leq 1.$$

Here $s(\mu)$ is a function introduced for convenience of notation and the J-dimensional column vector S has components $S_j = s(\mu_j)$.

To obtain the desired relation we first simplify (10.20) by noting, from (8.8), (7.5b), (7.8) and (7.3) with $n = 0$, that

$$(10.23) \qquad \begin{aligned} \underset{\rightarrow}{\omega}\,(m^2K^2) &= \underset{\rightarrow}{\omega}\,(I - c\,\Omega) \, , \\ &= (1-c)\,\underset{\rightarrow}{\omega} \quad . \end{aligned}$$

From (10.17) we have, using the notation (10.21),

$$(10.24) \qquad (m^2K^2)^{-1}(\mathcal{S}_+ + \mathcal{S}_-) = S + \frac{c}{1-c}\,[\underset{\rightarrow}{\omega} \circ S]\,\underset{\sim}{1}.$$

We note that $[\underset{\rightarrow}{\omega} \circ S]$ is a scalar and we have introduced the J-dimensional column vector

$$(10.25) \qquad \underset{\sim}{1} \equiv \begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}.$$

Using (10.21), (10.23) and (10.24) in (10.20) we obtain

$$(10.26) \qquad p_J(S) = \frac{1-c}{\sigma_T a} \frac{\underset{\rightarrow}{\omega}\, QS}{\underset{\rightarrow}{\omega}\, S} + \frac{c}{\sigma_T a} \underset{\rightarrow}{\omega}\; Q\, \underset{\sim}{1} .$$

Here we have included the argument S in p_J to explicitly indicate the dependence on the angular distribution of the source.

With similar notation, from (10.21), the capture fraction (10.15) can be written as

$$(10.27) \qquad \begin{aligned} f_J(S) &= 2(1-c) \frac{\underset{\rightarrow}{\omega}\, QS}{\underset{\rightarrow}{\omega}\, mS} , \\ &= \frac{2(1-c)}{\bar{\mu}_J(S)} \frac{\underset{\rightarrow}{\omega}\, QS}{\underset{\rightarrow}{\omega}\, S} . \end{aligned}$$

We have introduced here the source-weighted average cosine

$$(10.28)(a) \qquad \bar{\mu}_J(S) \equiv \frac{\underset{\rightarrow}{\omega}\, mS}{\underset{\rightarrow}{\omega}\, S} .$$

From the definitions of $\underset{\rightarrow}{\omega}$, m and S it is seen that this is a numerical approximation of

$$(10.28)(b) \qquad \bar{\mu}(s) \equiv \frac{\int_0^1 \mu s(\mu) d\mu}{\int_0^1 s(\mu) d\mu} \approx \bar{\mu}_J(S) .$$

Furthermore by (7.3) we see that $\bar{\mu}_J(S) = \bar{\mu}(s)$ if $s(\mu)$ is a polynomial in μ of degree $\leq N-1$. Of course as $J \rightarrow \infty$ in an appropriate manner $\bar{\mu}_J(S) \rightarrow \bar{\mu}(s)$ for all integrable functions $s(\mu)$.

An isotropic source is represented by setting

$$S = \underset{\sim}{1}\,, \quad (\text{i.e. } s(\mu) = 1)\,.$$

Then from (10.27) for an isotropic source, and recalling by (7.3) that $\underset{\rightarrow}{\omega}\cdot\underset{\sim}{1} = 1$, we have

$$f_J(\underset{\sim}{1}) = \frac{2(1-c)}{\overline{\mu}_J(\underset{\sim}{1})}\ \underset{\rightarrow}{\omega}\ Q\,\underset{\sim}{1}\,.$$

Using this result and (10.27) in (10.26) we have

$$(10.29)\quad p_J(S) = \frac{\overline{\mu}_J(S)}{\sigma_T 2a}\, f_J(S) + \frac{c}{1-c}\,\frac{\overline{\mu}_J(\underset{\sim}{1})}{\sigma_T\, 2a}\, f_J(\underset{\sim}{1})\,.$$

As this relation holds for all meshes (7.0) (i.e. for all J) we may assume, following Chandrasekhar, that it is an exact relation between the actual escape probability and capture fractions. That is, in an obvious notation, we have

$$(10.30)\quad \boxed{P(s) = \frac{\overline{\mu}(s)}{\sigma_T 2a}\, F(s) + \frac{c}{1-c}\,\frac{\overline{\mu}(1)}{\sigma_T 2a}\, F(1)}\,.$$

The relation (10.30) may be verified in two special cases. First, if the incident and distributed sources are isotropic then $s(\mu) \equiv 1$ and from (10.28) $\overline{\mu}(1) = 1/2$, and (10.30) becomes

$$(10.31)\quad \begin{aligned} P(1) &= \frac{F(1)}{2(1-c)\sigma_T 2a}\,, \\ &= \frac{F(1)}{2(\sigma_T - \sigma_s)2a}\,. \end{aligned}$$

Here we have used (6.1a) and the result, (10.31), is well

known [14]. In the second case consider purely absorbing media in which $c = 0$ and $\sigma_T = \sigma_a$, the absorption cross section. Then (10.30) becomes simply

$$P(s) = \frac{\bar{\mu}(s)}{\sigma_a 2a} F(s) . \tag{10.32}$$

This result is generally known for arbitrary convex bodies provided the sources are isotropic [15]. In the form (10.32) we have a generalization to arbitrary sources but which is valid only for slabs.

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